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Journal of Approximation Theory 134 (2005) 1–10

Journal of Approximation Theory
www.elsevier.com/locate/jat

Universal Taylor series on doubly connected domains with respect to every center

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Received 11 June 2003; accepted 10 September 2004

Communicated by András Kroó

Abstract

We prove that there exist universal Taylor series -in the sense of Nestoridis- in the complement of a square with respect to every center. Furthermore, for a weaker notion of universal Taylor series due to Luh and Chui and Parnes, we prove that there exist universal Taylor series in the complement of the closed unit disk with respect to every center. Overconvergence phenomena with respect to different centers have been first investigated by W. Luh (Analysis 6 (1986) 191–207).

MSC: 30B

Keywords: Taylor series; Baire's theorem; Runge's theorem

1. Introduction

Let Ω be an open subset of the complex plane. For a holomorphic function f in Ω ($f \in H(\Omega)$) and $\zeta \in \Omega$, we denote by $S_n(f, \zeta)$ the nth partial sum of the Taylor development of f, with center ζ, i.e.,

$$S_n(f,\zeta) = \sum_{k=0}^{n} \frac{f^{(k)}(\zeta)}{k!}(z-\zeta)^k.$$

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doi:10.1016/j.jat.2004.09.003

In a disk centered at ζ with radius R, $(D(\zeta, R))$, such that its closure is contained in Ω, the partial sums converge to the function f uniformly on $\overline{D(\zeta, R)}$. However outside Ω, subsequences of partial sums may have certain approximation properties. In this case we say that the sequence of partial sums overconverges. The next two definitions have been given in [7], see also [9,10].

Definition 1.1. Let Ω be an open set and $\zeta \in \Omega$. A function $f \in H(\Omega)$ belongs to the class $U(\Omega, \zeta)$, if for every compact set $K \subset \mathcal{C} \setminus \Omega$ with K^c connected and for every function $h : K \to C$, continuous on K and holomorphic in K^o, there exists a sequence $\{\lambda_n\}$ of natural numbers such that

$$\sup_{z\in K} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \to 0$$

as $n \to +\infty$.

Definition 1.2. A function $f \in H(\Omega)$ belongs to the class $U(\Omega)$ if for every K, h as in Definition 1.1, there is a sequence $\{\lambda_n\}$ of natural numbers such that for every $L \subset \Omega$ compact the following holds:

$$\sup_{\zeta\in L} \sup_{z\in K} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \to 0$$

as $n \to +\infty$.

The space $H(\Omega)$ of holomorphic functions in Ω becomes a complete metric space when endowed with the topology of uniform convergence on compact subsets of Ω.

Elements of the class $U(\Omega, \zeta)$ are called universal Taylor series with respect to ζ, in the sense that the partial sums $S_n(f, \zeta)$ approximate “everything we can hope for” outside Ω. Let me mention that in the early 70s independently Luh [4] and Chui and Parnes [2], gave a similar definition, where the compact set K is not allowed to contain pieces of the boundary of Ω and we denote this class by $U_1(\Omega, \zeta)$. This restriction produces many differences between the two classes $U(\Omega, \zeta)$, $U_1(\Omega, \zeta)$ see [8]. Observe that $U(\Omega, \zeta) \subset U_1(\Omega, \zeta)$. Elements of the class $U(\Omega)$ are also called universal Taylor series. We may also consider the class $U_1(\Omega)$ if in Definition 1.2, instead of $K \bigcap \Omega = \emptyset$, the set K satisfies the property $K \bigcap \overline{\Omega} = \emptyset$. An immediate consequence is that $U(\Omega) \subset U_1(\Omega)$.

Obviously $U(\Omega) \subset U(\Omega, \zeta)$. However both classes are not always non-empty. Actually the existence of universal Taylor series on some open set Ω depends on the set Ω itself. Let me briefly mention, in this direction, the following known results:

(1) If Ω is a simply connected domain, both classes $U(\Omega, \zeta)$, $U(\Omega)$ are G_δ and dense in $H(\Omega)$ and if in addition Ω is contained in the complement of a positive angle then $U(\Omega, \zeta) = U(\Omega)$ for all $\zeta \in \Omega$, see [3,5,7].

(2) If Ω is a non-simply connected domain then always $U(\Omega) = \emptyset$ and if Ω is also contained in the complement of a positive angle then $U(\Omega, \zeta) = \emptyset$, see [3,7].

However, there are non-simply connected domains which support universal Taylor series with respect to one center. For example if K is a connected compact set and also $\mathcal{C} \setminus K$ is connected, then for $\Omega = \mathcal{C} \setminus K$ and $\zeta \in \Omega$, the class $U(\Omega, \zeta)$ is G_δ and dense in $H(\Omega)$, thus non-empty, see [6] (see also [11] when K is a singleton). Also, in case Ω is any simply connected domain then, as it is remarked above, $U(\Omega)$ is G_δ and dense in $H(\Omega)$, hence $\bigcap_{\zeta\in\Omega} U(\Omega,\zeta)$ is residual (since $U(\Omega)\subset\cap_{\zeta\in\Omega}U(\Omega,\zeta)$), but it is not known in general if $U(\Omega,\zeta_1)=U(\Omega,\zeta_2)$ for all $\zeta_1,\zeta_2\in\Omega$.

In [6], Melas proved the following interesting result: there is a non-simply connected domain Ω with $0\in D\subset\Omega$, (by D we denote the open unit disk), $\mathcal{C}\setminus\Omega$ is infinite and discrete such that $U(\Omega,0)\neq U(\Omega,\zeta)$ for every $\zeta\in\Omega\setminus D\neq\emptyset$. However, we do not know if $\cap_{\zeta\in\Omega} U(\Omega,\zeta)$ is non-empty.

So far it is not known if there is a non-simply connected domain Ω such that $\bigcap_{\zeta\in\Omega} U(\Omega,\zeta)\neq\emptyset$. The purpose of the present work is to provide a class of non-simply connected domains Ω, for which the uncountable intersections of classes of universal Taylor series give a residual set of universal Taylor series with respect to any center $\zeta\in\Omega$. More precisely we prove the following:

Theorem 1.3. *Let K be a closed square with its interior and $\Omega=\mathcal{C}\setminus K$. Then, although $U(\Omega)=\emptyset$ see* [7], *the class $\bigcap_{\zeta\in\Omega} U(\Omega,\zeta)$ is residual in $H(\Omega)$, hence non-empty.*

Actually we can extend Theorem 1.3 for every K, where K is a closed polygonal line with its interior. However we prefer to state Theorem 1.3 for a square, because the proof is more transparent. After the proof of Theorem 1.3 is presented, we sketch the proof for the general case of a polygon. Unfortunately, we were unable to prove a corresponding result for the complement of the closed unit disk and so we ask the following: is it true that $\bigcap_{\zeta\in\mathcal{C}\setminus\overline{D}} U(\mathcal{C}\setminus\overline{D},\zeta)$ is residual in $H(\mathcal{C}\setminus\overline{D})$, thus non-empty?

We are able to answer in the affirmative way the above question if we replace the class $U(\mathcal{C}\setminus\overline{D},\zeta)$ with the weaker class $U_1(\mathcal{C}\setminus\overline{D},\zeta)$, where the compact set K, in which the approximation takes place, doesn't contain pieces of the unit circle. So we establish the next theorem.

Theorem 1.4. *Let D be the open unit disk. Then, although $U_1(\mathcal{C}\setminus\overline{D})=\emptyset$ see* [7], *the class $\bigcap_{\zeta\in\mathcal{C}\setminus\overline{D}} U_1(\mathcal{C}\setminus\overline{D},\zeta)$ is residual in $H(\Omega)$, hence non empty.*

2. Proof of Theorem 1.3

Let $K=\{z=x+iy : -1\leqslant x\leqslant 1,\ -1\leqslant y\leqslant 1\}$ and let Ω be the complement of K i.e. $\Omega=\mathcal{C}\setminus K$. From now on, K and Ω are fixed.

For the clarity of proof it is convenient to use the following definition, which has been taken from [7].

Definition 2.1. Let $L\subset\Omega$. We say that a function f, holomorphic in Ω belongs to the class $U(\Omega,L)$ if for every function $h:K\to C$, continuous on K and holomorphic in K^o, there exists a sequence $\{\lambda_n\}$ of natural numbers such that

$$\sup_{\zeta\in L}\sup_{z\in K}|S_{\lambda_n}(f,\zeta)(z)-h(z)|\to 0$$

as $n\to+\infty$.

At this point we would like to comment on the similarities and differences between the classes $U(\Omega)$ (see Definition 1.2) and $U(\Omega, L)$. The basic difference is that when we deal with the class $U(\Omega)$, the approximative sequence $\{\lambda_n\}$ is the same for all the centers lying on any compact set $L \subset \Omega$; therefore the sequence $\{\lambda_n\}$ depends only on the compact set K and the function h. On the other hand, dealing with the class $U(\Omega, L)$, it is obvious that the sequence $\{\lambda_n\}$ depends on the compact set L (the set where the centers are lying) as well. However as we shall see below, if we impose certain topological and geometrical restrictions on the set Ω then the classes coincide. Let Ω be any open set, $L \subset \Omega$ be a compact set and $\zeta \in L$. By only using the definitions of the classes $U(\Omega)$, $U(\Omega, \zeta)$, $U(\Omega, L)$ (observe that Definitions 1.1, 1.2 and 2.1 can be given for any open set Ω) we have

$$U(\Omega) \subset U(\Omega, L) \subset U(\Omega, \zeta).$$

If Ω is a simply connected domain which is contained in the complement of a positive angle and because of the previous inclusion and (1) (see Introduction), we conclude that for any compact set $L \subset \Omega$, the classes $U(\Omega)$, $U(\Omega, L)$ coincide. The situation in non-simply connected domains turns out to be rather different. It is known that for any non-simply connected domain Ω the class $U(\Omega)$ is empty. On the other hand, we shall show below that for Ω being the complement of a closed square and for certain compact sets $L \subset \Omega$, the class $U(\Omega, L)$ is G_δ and dense in $H(\Omega)$.

Let us now proceed with the proof of Theorem 1.3. Consider the four (closed) quadrants of the plane and then take the intersection of each one with Ω, that is

$$\Omega_1 = \Omega \cap \{z = x + iy : x \geqslant 0,\ y \geqslant 0\},$$

$$\Omega_2 = \Omega \cap \{z = x + iy : x \leqslant 0,\ y \geqslant 0\},$$

$$\Omega_3 = \Omega \cap \{z = x + iy : x \leqslant 0,\ y \leqslant 0\},$$

$$\Omega_4 = \Omega \cap \{z = x + iy : x \geqslant 0,\ y \leqslant 0\}.$$

For every $\mu = 1, 2, 3, 4$, consider a sequence of compact sets L_ρ^μ of Ω_μ such that

$$\Omega_\mu = \cup_{\rho=1}^{\infty} L_\rho^\mu.$$

In particular, we may define the compact sets L_ρ^μ as follows.

$$L_\rho^\mu = \Omega_\mu \bigcap \{x + iy : |x| \leqslant \rho,\ |y| \leqslant \rho\} \bigcap \left\{x + iy : |x| < 1 + \frac{1}{\rho},\ |y| < 1 + \frac{1}{\rho}\right\}^c$$

for $\mu = 1,\ 2,\ 3,\ 4$ and $\rho = 1,\ 2,\ 3, \ldots$.

Take a countable collection of all polynomials with coefficients in $Q + iQ$ and consider an enumeration of them, $f_1,\ f_2, \ldots$. Let us define the set

$$E(j, s, n, \rho, \mu) = \left\{ f \in H(\Omega) : \sup_{\zeta \in L_\rho^\mu} \sup_{z \in K} |S_n(f, \zeta)(z) - f_j(z)| < \frac{1}{s} \right\}$$

for $j, s, \rho = 1, 2, 3, \ldots$, $n = 0, 1, 2, \ldots$ and $\mu = 1, 2, 3, 4$.

Now, it is not difficult to prove, see [9,7], the following.

Lemma 2.2. (1) $U(\Omega, L_\rho^\mu) = \bigcap_j \bigcap_s \bigcup_n E(j,s,n,\rho,\mu)$ *and*
(2) $E(j,s,n,\rho,\mu)$ *is open in* $H(\Omega)$
for every $\mu = 1, 2, 3, 4$ *and* $\rho = 1, 2, 3, \ldots$.

Observe that if the set $\bigcup_n E(j,s,n,\rho,\mu)$ is dense in $H(\Omega)$, then because of Lemma 2.2 and Baire's category theorem, we obtain that the set $U(\Omega, L_\rho^\mu)$ is G_δ and dense in $H(\Omega)$, for every $\mu = 1, 2, 3, 4$ and $\rho = 1, 2, 3, \ldots$. At this point let us see how we can finish the proof of Theorem 1.3, provided that $U(\Omega, L_\rho^\mu)$ is G_δ and dense in $H(\Omega)$. Actually, it is enough to observe that:

(1) $\bigcap_{\mu=1}^4 \bigcap_{\rho=1}^\infty U(\Omega, L_\rho^\mu)$ is G_δ and dense in $H(\Omega)$, as countable intersection of G_δ and dense sets and

(2) $\bigcap_{\mu=1}^4 \bigcap_{\rho=1}^\infty U(\Omega, L_\rho^\mu) \subset \bigcap_{\mu=1}^4 \bigcap_{\zeta\in\Omega_\mu} U(\Omega,\zeta) = \bigcap_{\zeta\in\Omega} U(\Omega,\zeta)$.

Thus, it only remains to prove the following.

Lemma 2.3. *For every* $\mu = 1, 2, 3, 4$ *and every* $j, s, \rho \geqslant 1$ *the set* $\bigcup_n E(j,s,n,\rho,\mu)$ *is dense in* $H(\Omega)$.

Proof. Let $f \in H(\Omega)$, $L \subset \Omega$ compact and $\varepsilon > 0$. We look for a function $g \in H(\Omega)$ and a natural number $n \geqslant 0$ so that

$$\sup_{z\in L} |f(z) - g(z)| < \varepsilon \tag{1}$$

and

$$\sup_{\zeta\in L_\rho^\mu} \sup_{z\in K} |S_n(g,\zeta)(z) - f_j(z)| < \frac{1}{s}. \tag{2}$$

We may consider without loss of generality that $\mu = 1$. By Runge's theorem there exists a rational function ϕ having no pole other than at $w_1 \in \Omega \setminus (L \cup K)$, such that

$$\sup_{z\in L} |f(z) - \phi(z)| < \frac{\varepsilon}{2} \tag{3}$$

and

$$\sup_{z\in K} |f_j(z) - \phi(z)| < \frac{1}{3s}. \tag{4}$$

Observe that for every $\zeta \in L_\rho^1$ and choosing a point w_1 in the bounded connected component of $\Omega \setminus (L \cup K)$ (since $L \subset \Omega$ is compact, we may assume that $\Omega \setminus L$ has a bounded connected component V such that $K \subset V$) so that, w_1 belongs to the line joining $1+i$, $-1-i$, we have

$$\sup_{z\in K} |S_n(\phi,\zeta)(z) - \phi(z)| \to 0 \tag{5}$$

as $n \to \infty$.

We want to replace the previous limit, see (5), by the uniform limit for all $\zeta \in L_\rho^1$. In order to do that, we shall make the final choice for w_1.

Let us define

$$R_\zeta = \sup_{z\in K} |z-\zeta| + \delta, \tag{6}$$

where δ is chosen such that for every $\zeta \in L^1_\rho$ we have

$$K \subset D(\zeta, R_\zeta) \quad \text{and} \quad w_1 \notin \overline{D(\zeta, R_\zeta).} \tag{7}$$

By using Cauchy's estimates and (6), (7) we obtain that

$$\begin{aligned} &\sup_{\zeta\in L^1_\rho} \sup_{z\in K} |S_n(\phi,\zeta)(z) - \phi(z)| \\ &\quad \leqslant \sum_{m=n+1}^{\infty} \sup_{\zeta\in L^1_\rho} \sup_{|w-\zeta|\leqslant R_\zeta} |\phi(w)| \sup_{\zeta\in L^1_\rho} \frac{\sup_{z\in K} |z-\zeta|^m}{{R_\zeta}^m}. \end{aligned} \tag{8}$$

The last term in the previous inequality tends to 0 as $n \to \infty$, since we can easily see that there is $0 < \theta < 1$ such that

$$\sup_{\zeta\in L^1_\rho} \frac{\sup_{z\in K} |z-\zeta|}{R_\zeta} < \theta$$

and thus the series in (8) is dominated by a geometric one. From the above we conclude that $\sup_{\zeta\in L^1_\rho} \sup_{z\in K} |S_n(\phi,\zeta)(z) - \phi(z)| \to 0$ as $n \to \infty$, so we can fix n such that

$$\sup_{\zeta\in L^1_\rho} \sup_{z\in K} |S_n(\phi,\zeta)(z) - \phi(z)| < \frac{1}{3s}. \tag{9}$$

Take any point $z_1 \in K^o$ and let R be such that

$$\overline{D(\zeta, R)} \bigcap K = \emptyset \tag{10}$$

for every $\zeta \in L^1_\rho$.

Let us also fix a positive number $\varepsilon_1 > 0$ such that

$$\varepsilon_1 < \min\left\{ \frac{\varepsilon}{2}, \frac{1}{3s \sum_{m=0}^{n} \frac{\sup_{\zeta\in L^1_\rho} \sup_{z\in K} |z-\zeta|^m}{R^m}} \right\}. \tag{11}$$

By Runge's theorem and (10) we can find a rational function g having a pole at z_1, satisfying

$$\sup_{z\in L \bigcup \cup_{\zeta\in L^1_\rho} \overline{D(\zeta,R)}} |\phi(z) - g(z)| < \varepsilon_1. \tag{12}$$

Cauchy's estimates imply that

$$\sup_{\zeta\in L^1_\rho}\sup_{z\in K}|S_n(\phi,\zeta)(z)-S_n(g,\zeta)(z)| \leqslant \sup_{\zeta\in L^1_\rho}\sup_{|w-\zeta|\leqslant R}|\phi(w)-g(w)|\sum_{m=0}^{n}\frac{\sup_{\zeta\in L^1_\rho}\sup_{z\in K}|z-\zeta|^m}{R^m}. \tag{13}$$

Combining (9), (11) and (13) we get

$$\sup_{\zeta\in L^1_\rho}\sup_{z\in K}|S_n(g,\zeta)(z)-\phi(z)|<\frac{2}{3s}. \tag{14}$$

From relations (4) and (14) it is straightforward that

$$\sup_{\zeta\in L^1_\rho}\sup_{z\in K}|S_n(g,\zeta)(z)-f_j(z)|<\frac{1}{s}. \tag{15}$$

Finally, (3), (12) and (13) imply

$$\sup_{z\in L}|f(z)-g(z)|<\varepsilon. \tag{16}$$

Since $g\in H(\Omega)$ and because of (15), (16) the result follows. This completes the proof of Lemma 2.3. Thus the proof of Theorem 1.3 is finished. □

Now we would like to comment on the main idea of the proof and how our method can be extended to the case of the complement of a polygon. The crucial step in our approach is the division of the domain Ω into four regions Ω_i, i=1, 2, 3, 4 such that:

the maximum distance between K and every compact $L^i_\rho\subset\Omega_i$, i.e. $\max\{|z-w| : z\in K,\ w\in L^i_\rho\}$ is attained on $z(K)$, $w(L^i_\rho)$, where $z(K)$ is the same for all compact subsets of Ω_i. Actually $z(K)$ is exactly one of the four vertices of the square and of course $z(K)$ depends only on the domain Ω_i, i=1,2,3,4.

After that, we are allowed to chose the pole of the rational function ϕ appropriately, so that we can control the quantity $\sup_{z\in K}|S_n(\phi,\zeta)(z)-\phi(z)|$ uniformly for all $\zeta\in L^i_\rho$ as $n\to\infty$.

Let us now sketch briefly the crucial step of the proof of a generalization of Theorem 1.3, in case we replace the square with a polygon having n vertices. For every two vertices consider the corresponding segment joining the two vertices. For every such segment, take its middle point and draw the line which is perpendicular to the segment and passing through the middle point. The collection of these lines divides the complement of the polygon into some "regions". For each one "region" consider an exhaustive family of compact sets. We have to observe that for every compact set L, which we select from the same exhaustive family, the maximum distance between L and the polygon is always attained at the same vertex of the polygon. Of course, to different "regions" there correspond different vertices. After that, the proof for the case of a polygon follows the lines of the proof of Theorem 1.3 with minor modifications and the details are left to the reader.

3. Proof of Theorem 1.4

Fix $\{\zeta_\rho\}$ a countable dense set in $\overline{D}^c$. For every $m = 1, 2, 3\ldots$ we define the countable set

$$\{\zeta_\rho^m\} = \{\zeta_\rho\} \bigcap \left\{ \zeta \in \overline{D}^c : |\zeta| > 1 + \frac{1}{2m-1} \right\}.$$

Observe that

$$\bigcup_{m=1}^{\infty} \bigcup_{\rho=1}^{\infty} \overline{D\left(\zeta_\rho^m, \frac{1}{2m+1}\right)} = \mathcal{C} \setminus \overline{D}. \tag{17}$$

Let us define $K_m = \{z : |z| \leqslant 1 - \frac{1}{m}\}$, for $m = 1, 2, 3\ldots$.

Definition 3.1. Consider any $L \subset \mathcal{C} \setminus \overline{D}$ compact. We say that a holomorphic function $f \in H(\mathcal{C} \setminus \overline{D})$ belongs to the class $U_1(\mathcal{C} \setminus \overline{D}, K_m, L)$ if and only if for every $h : K_m \to \mathcal{C}$, continuous on K_m and holomorphic in K_m^o there is a sequence $\{\lambda_n\}$ of natural numbers such that

$$\sup_{\zeta \in L} \sup_{z \in K_m} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \to 0$$

as $n \to +\infty$.

We want to show that the set $\bigcap_{\zeta \in \mathcal{C} \setminus \overline{D}} U_1(\mathcal{C} \setminus \overline{D}, \zeta)$ is residual in $H(\mathcal{C} \setminus \overline{D})$. For that, it is enough to prove the following:

Lemma 3.2. (i) *The following inclusion holds.*

$$\bigcap_{m=1}^{\infty} \bigcap_{\rho=1}^{\infty} U_1\left(\mathcal{C} \setminus \overline{D}, K_m, \overline{D\left(\zeta_\rho^m, \frac{1}{2m+1}\right)}\right) \subset \bigcap_{\zeta \in \mathcal{C} \setminus \overline{D}} U_1(\mathcal{C} \setminus \overline{D}, \zeta).$$

(ii) *The set* $\bigcap_{m=1}^{\infty} \bigcap_{\rho=1}^{\infty} U_1(\mathcal{C} \setminus \overline{D}, K_m, \overline{D(\zeta_\rho^m, \frac{1}{2m+1})})$ *is* G_δ *and dense in* $H(\mathcal{C} \setminus \overline{D})$.

Let us remark that, because of (17) the above inclusion is obviously true. So it only remains to prove part (ii) of Lemma 3.2. In view of Baire's theorem, part (ii) of Lemma 3.2 will be true, if the following holds.

Lemma 3.3. *For every* $m = 1, 2, 3\ldots$ *fix any* $\zeta(m) = \zeta$ *such that* $|\zeta| > 1 + \frac{1}{2m-1}$. *Then for every* $m = 1, 2, 3\ldots$ *the set* $U_1(\mathcal{C} \setminus \overline{D}, K_m, \overline{D(\zeta, \frac{1}{2m+1})})$ *is* G_δ *and dense in* $H(\mathcal{C} \setminus \overline{D})$.

Proof. Take an enumeration of the polynomials f_j with coefficients in $Q + iQ$, and define the set

$$E(m, j, s, n) = \left\{ g \in H(\mathcal{C} \setminus \overline{D}) : \sup_{w \in \overline{D(\zeta, \frac{1}{2m+1})}} \sup_{z \in K_m} |S_n(g, w)(z) - f_j(z)| < \frac{1}{s} \right\},$$

for every $m, j, s \geqslant 1$ and $n \geqslant 0$.

Then, it is standard to prove that

$$U_1\left(\mathcal{C}\setminus\overline{D}, K_m, \overline{D\left(\zeta, \frac{1}{2m+1}\right)}\right) = \bigcap_{j=1}^{\infty}\bigcap_{s=1}^{\infty}\bigcup_{n=0}^{\infty} E(m, j, s, n)$$

and that $E(m, j, s, n)$ is open in $H(\mathcal{C}\setminus\overline{D})$, see [7]. Thus, because of Baire's category theorem and in order to finish the proof of Lemma 3.3, it suffices to prove that for every $j, s = 1, 2, \ldots,$ the set $\bigcup_{n=0}^{\infty} E(m, j, s, n)$ is dense in $H(\mathcal{C}\setminus\overline{D})$.

For that, fix $f \in H(\mathcal{C}\setminus\overline{D})$, $j, s \in \{1, 2, \ldots\}$ and consider any compact set $L \subset \mathcal{C}\setminus\overline{D}$ and $\varepsilon > 0$. We look for a function $g \in H(\mathcal{C}\setminus\overline{D})$ and a $n \in N$ such that

$$\sup_{z\in L}|f(z) - g(z)| < \varepsilon \tag{18}$$

and

$$\sup_{w\in\overline{D(\zeta, \frac{1}{2m+1})}}\ \sup_{z\in K_m} |S_n(g, w)(z) - f_j(z)| < \frac{1}{s}. \tag{19}$$

Let us define the following function:

$$h(z) = f_j(z),\ \ z \in K_m,$$

$$h(z) = f(z),\ \ z \in L.$$

Fix the point w_o on the unit circle so that w_o is the intersection of the unit circle with the line joining ζ with 0.

By using Runge's theorem we can approximate h on $K_m \cup L$ by a rational function g with no pole other than at w_o such that

$$\sup_{z\in L}|f(z) - g(z)| < \varepsilon \tag{20}$$

and

$$\sup_{z\in K_m} |g(z) - f_j(z)| < \frac{1}{2s}. \tag{21}$$

We turn our attention to the difference $g(z) - S_n(g, w)(z)$, which we want to estimate for $w \in \overline{D(\zeta, \frac{1}{2m+1})}$. Cauchy estimates and the fact that for every $w \in \overline{D(\zeta, \frac{1}{2m+1})}$ the point w_o is not contained in the closed disk $|w - z| \leqslant |\zeta| + 1 - \frac{1}{2m}$, imply that

$$\begin{aligned}&\sup_{w\in\overline{D(\zeta, \frac{1}{2m+1})}}\ \sup_{z\in K_m} |g(z) - S_n(g, w)(z)|\\ &\quad\leqslant \sup_{w\in\overline{D(\zeta, \frac{1}{2m+1})}}\ \sup_{|w-z|\leqslant|\zeta+1-\frac{1}{2m}} |g(z)| \sum_{k=n+1}^{\infty}\delta^k,\end{aligned} \tag{22}$$

where $\delta := \frac{|\zeta|+1-\frac{1}{m}}{|\zeta|+1-\frac{1}{2m}}$ and $0 < \delta < 1$.

From the above we may choose n sufficiently large so that the last term in (22) is less than $\frac{1}{2s}$ and fix such an n. Then, using relations (20), (21), the approximation properties (18) and (19) are satisfied. This completes the proof of Lemma 3.3 and hence that of Theorem 1.4. □

Remark 3.4. In a recent paper, Bayart [1] answered our question about the existence of universal Taylor series in the sense of Nestoridis in $\mathbb{C} \setminus \overline{D}$ with respect to every center $\zeta \in \mathbb{C} \setminus \overline{D}$. In fact he proved that the class $\cup_{\zeta \in \mathbb{C} \setminus \overline{D}} U(\mathbb{C} \setminus \overline{D}, \zeta)$ is residual in $H(\mathbb{C} \setminus \overline{D})$. His main idea is to approximate the unit circle by suitable polygonal lines and at the same time he obtains an approximative sequence $\{\lambda_n\}$ with controlled growth for centers lying on certain compact sets.

Acknowledgments

I would like to thank the referees for a very careful reading of the manuscript and for their valuable suggestions.

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